ON THE EXISTENCE OF A SOLUTION TO THE PROBLEM OF THE EQUILIBRIUM OF A CIRCULAR MEMBRANE

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It is shown that a solution exists to the problem of the equilibrium of a symmetrically loaded membrane with an unstressed contour. The proof uses the method of Chaplygin and at the same time a numerical method of solution is derived.

Let us consider the problem of the equilibrium of a circular, symmetrically loaded membrane with an unstressed contour [1]

$$Lv \equiv -\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho v = \frac{\varphi^2}{2\rho v^2}, \qquad \varphi(\rho) = \int_0^\rho q(t)\, t\, dt \tag{1}$$

$$\frac{v}{\rho}\Big|_{\rho=0} < \infty, \qquad v|_{\rho=1} = 0 \tag{2}$$

Here the function v corresponds to the radial stress and $q(\rho)$ is the intensity of normal loading. In the case when $\varphi(\rho)$ satisfies the condition $\varphi(1) = 0$, the existence theorem was formulated without proof in [2], and in the case of the load $\varphi(\rho) = q\rho^2$ ($q = \mathrm{const}$) uniformly distributed over the surface the existence was proved in [3]. Chaplygin's method [4 and 5] enables us to derive an effective method for constructing the solution (see formulas (5) to (8) below), while proving its existence. It should be noted that in constrast to the previously employed method of power series [1], the function $\varphi(\rho)$, needs not to be analytic, and this allows us to analyse a membrane under the action of discontinuous loads.

Theorem. Suppose that the function $\varphi(\rho)$ is piecewise continuous. Then the problem (1) and (2) has a continuous solution and the following inequalities hold for the derivatives v' and v'':

$$|v'(\rho)| \leqslant m_1 (1-\rho)^{-1/3}, \qquad |v''(\rho)| \leqslant m_2 (1-\rho)^{-4/3} \tag{3}$$

(here and in the following m_i are the constants independent of ρ).

Proof. The problem (1) and (2) is equivalent to the integral equation

$$v = L^{-1}\left(\frac{\varphi^2}{2\rho v^2}\right) \qquad \left(L^{-1}f = \rho\int_\rho^1 t^{-3}\,dt\int_0^t f(\tau)\,\tau^2\,d\tau\right) \tag{4}$$

We can show that the solution of the problem will be the limit of a sequence of functions $\{v_n\}$ defined by the relations

$$v_{n+1} = v_n - \delta_n \qquad (n = 1, 2, \ldots) \tag{5}$$

$$v_1 = \frac{1}{2C^2}L^{-1}\left(\frac{\varphi^2}{\rho^2(1-\rho)^{4/3}}\right), \qquad C = \max_{0\leqslant\rho\leqslant 1}\left[\frac{15}{4}\,\frac{\varphi^2}{\rho^4}\right]^{1/3} \tag{6}$$

$$L\delta_n + \frac{M}{(1-\rho)^2}\delta_n - \alpha_n = 0, \qquad \frac{\delta_n}{\rho}\Big|_{\rho=0} < \infty, \qquad \delta_n(1) = 0 \tag{7}$$

$$\alpha_n = L\delta_n - \frac{\varphi^2}{2\rho v_n^2}, \qquad M = \max_{0\leqslant\rho\leqslant 1}\left|\frac{\varphi^2(\rho)(1-\rho)^2}{\rho v_1^3}\right| \tag{8}$$

The quantity M is finite, since from the condition that $|\varphi(\rho)| \leqslant m_3\rho^2$, and from (4) and (6) it follows that

$$C\rho\,(1-\rho)^{2/3} \geqslant v_1 \geqslant m_4\rho\,(1-\rho)^{2/3} \qquad (m_4 > 0) \tag{9}$$

if $\varphi(\rho) \not\equiv 0$ for $0 \leqslant \rho \leqslant 1$.

Indeed, suppose we find an interval $[a, b] \subset [0, 1]$, in which $\min \varphi^2/\rho^4 = m_1 > 0$, and suppose, for instance, that $b = 1$. Then from (4) and (6) we have

$$v_1 \geqslant \frac{1}{2C^2}\rho\int_\rho^1 dt\int_a^t \frac{\varphi^2}{\tau^4(1-\tau)^{4/3}}\,d\tau \geqslant m_5\rho J(\rho), \qquad J(\rho) = \int_\rho^1 f(t)\,dt$$
$$f(t) = \begin{cases} 0 & (0 \leqslant t \leqslant a) \\ [(1-t)^{-1/3} - (1-a)^{-1/3}] & (a \leqslant t \leqslant 1)\end{cases} \tag{10}$$

From this we easily obtain

$$v_1(\rho) \geqslant J(\rho) = \begin{cases} m_5\rho\,(1-a)^{2/3} & (\rho \leqslant a) \\ m_5\rho\,(1-\rho)^{2/3}\left[\dfrac{3}{2} - \left(\dfrac{1-\rho}{1-a}\right)^{1/3}\right] \geqslant \dfrac{m_5}{4}\rho\,(1-\rho)^{2/3} & (\rho \geqslant a)\end{cases}$$

From (8) and (9) we find that

$$\alpha_1 = Lv_1 - \frac{\varphi^2}{2\rho v_1^2} = \frac{\varphi^2\,[v_1^2 - C^2\rho^2(1-\rho)^{4/3}]}{2C^2\rho^3(1-\rho)^{4/3}v_1^2} \leqslant 0 \tag{11}$$

Next we shall prove that $\delta_1(\rho) \leqslant 0$. To do so we multiply (7) with $n = 1$ by δ_1, and integrate it with respect to ρ from 0 to 1. As a result we obtain

$$\int_0^1\left|\frac{d\delta_1}{d\rho}\right|^2 d\rho + \frac{1}{2}\int_0^1\frac{\delta_1^2}{\rho^2}\,d\rho + M\int_0^1\frac{\delta_1^2}{(1-\rho)^2}\,d\rho = \int_0^1\alpha_1\delta_1\,d\rho \tag{12}$$

If we estimate the left-hand side of (12) using the inequality

$$\delta^2(1) = \left(\int_0^1\frac{d\delta}{d\rho}\,d\rho\right)^2 \leqslant \int_0^1\left|\frac{d\delta}{d\rho}\right|^2 d\rho \tag{13}$$

applied to δ_1 we deduce that

$$\int_0^1 \alpha_1\delta_1 d\rho \geqslant 0 \tag{14}$$

If we now suppose that $\delta_1(\rho)$ assumes positive values, we can find an interval $[\xi_1, \xi_2] \subset [0, 1]$, such that $\delta_1(\rho) \geqslant 0$, for $\rho \in [\xi_1, \xi_2]$ and $\delta_1(\xi_1) = -\delta(\xi_2) = 0$. But this leads to a contradiction since analogously to (14) we obtain

$$\int_{\xi_1}^{\xi_2} \alpha_1\delta_1\, d\rho \geqslant 0$$

We shall now introduce the following function spaces:

(1) consisting of functions which satisfy conditions (2) and have a finite norm

$$(L_{2,\rho}) \qquad \|\delta\|_{L_{2,\rho}}^2 = \int_0^1 \left(\frac{\delta}{1-\rho}\right)^2 d\rho \tag{15}$$

(2) consisting of functions with the finite norm

$$(L_2{}^*) \qquad \|\alpha\|_{L_2^*}^2 = \int_0^1 (1-\rho)^2 \alpha^2\, d\rho \tag{16}$$

(3) obtained by the closure of the set of smooth functions given within $[0, 1]$, by the norm

$$\|f\|_{H_1}{}^2 = \int_0^1 \left|\frac{df}{d\rho}\right|^2 d\rho \tag{17}$$

Using the inequality $\|\delta\|_{L_{2,\rho}} \leqslant 2\|\delta\|_{H_1}$, we obtain from (7)

$$\|\delta_1\|_{L_{2,\rho}} \leqslant \frac{1}{M + {}^1/_4} \|\alpha_1\|_{L_2^*} \tag{18}$$

We can show that $\alpha_2 \leqslant 0$. We have that

$$\alpha_2 = Lv_2 - \frac{\varphi^2}{2\rho v_2{}^2} = \frac{\varphi^2}{2\rho v_1{}^2} - \frac{\varphi^2}{2\rho\,(v_1 - \delta_1)^2} + \frac{M}{(1-\rho)^2}\delta_1 \tag{19}$$

By means of Lagrange's formula we rewrite (19) in the form

$$\alpha_2 = \left[\frac{M}{(1-\rho)^2} - \frac{\varphi^2}{\rho\,(v_1 - \tau\delta_1)^3}\right]\delta_1 \qquad (0 \leqslant \tau \leqslant 1) \tag{20}$$

That α_2 cannot be positive follows from (20) on the basis of the definition (8) of M and the inequalities $v_1 \geqslant 0$ and $\delta_1 \leqslant 0$. Furthermore, from (20) it follows that $\|\alpha_2\|_{L_2^*} \leqslant M\|\delta_1\|_{L_{2,\rho}}$.

From (7) with $n = 2$, we obtain

$$\|\delta_2\|_{L_{2,\rho}} \leqslant \frac{1}{M + {}^1/_4} \|\alpha_1\|_{L_2^*} \tag{21}$$

Similarly we find that

$$\|\delta_k\|_{L_{2,\rho}} \leqslant \frac{1}{M + {}^1/_4} \|\alpha_k\|_{L_2^*}, \qquad \|\alpha_k\|_{L_2^*} \leqslant M\|\delta_{k-1}\|_{L_{2,\rho}} \tag{22}$$

whence for any $k \geqslant 1$ we obtain

$$\|\alpha_k\|_{L_2^*} \leqslant q^{k-1}\|\alpha_1\|_{L_2^*}, \quad \|\delta_k\|_{L_{2,\rho}} \leqslant \frac{q^{k-1}}{M+{}^1/_4}\|\alpha_1\|_{L_2^*}, \quad q = \frac{M}{M+{}^1/_4} \tag{23}$$

We shall now prove that the series $v_1 - (\delta_1 + \delta_2 + \delta_3 + \ldots)$, and therefore the sequence v_k, converge uniformly on $[0, 1]$ to some function v. From (7) we obtain the equation

$$\delta_k = L^{-1}\alpha_k - ML^{-1}\left(\frac{\delta_k}{(1-\rho)^2}\right) \tag{24}$$

Let us now evaluate $L^{-1}\alpha_k$. By using the Buniakovskii inequality, we obtain

$$|L^{-1}\alpha_k| \leqslant \frac{1}{2}\rho\|\alpha_k\|_{L_2^*}\int_\rho^1 t^{-3}\,dt\left[\int_0^t \frac{\tau^4}{(1-\tau)^2}\,d\tau\right]^{1/2} \tag{25}$$

If we apply the inequality $\tau < t$ to the inner integral and evaluate the resulting integral, we find from (25) that

$$|L^{-1}\alpha_k| \leqslant \frac{1}{2}\rho\|\alpha_k\|_{L_2^*}\int_\rho^1 t^{-1/2}(1-t)^{-1/2}\,dt \leqslant m_6\|\alpha_k\|_{L_2^*} \tag{26}$$

Similarly

$$\left|L^{-1}\left(\frac{\delta_k}{(1-\rho)^2}\right)\right| \leqslant m_7\|\delta_k\|_{L_{2,\rho}} \tag{27}$$

Now, from (24), (26) and (27) we obtain

$$\max_{0\leqslant\rho\leqslant1}|\delta_k\| \leqslant m_8(\|\alpha_k\|_{L_2^*} + \|\delta_k\|_{L_{2,\rho}}) \leqslant m_9 q^{k-1}\|\alpha_1\|_{L_{2,\rho}} \tag{28}$$

which confirms the convergence of the sequence v_k to v_0. It remains to show that v is the solution of (4). The following relation follows from (8):

$$v_k = L^{-1}\left(\frac{\varphi^2}{2\rho v_k^2}\right) + L^{-1}\alpha_k \tag{29}$$

The last term in (29) tends to zero as $k \to \infty$ on the basis of (26). Also, we note that

$$m_{10}\rho\,(1-\rho)^{2/3} \geqslant v \geqslant v_k \geqslant v_1 \geqslant m_3\rho\,(1-\rho)^{2/3} \tag{30}$$

The right-hand side of this inequality has already been proved and the left-hand side follows from the fact that

$$w \geqslant v \quad \text{for} \quad Lw - \frac{\varphi^2}{2\rho w^2} \geqslant 0$$

We can express w in the form

$$w = [{}^9/_2 \max\varphi^2(\rho)]^{1/3}\rho\,(1-\rho)^{2/3} \qquad (0 \leqslant \rho \leqslant 1)$$

From (30) we easily see that

$$\left|\frac{\varphi^2\rho}{v_k^2} - \frac{\varphi^2\rho}{v^2}\right| \leqslant \frac{m_{11}\rho^2}{(1-\rho)^2}\sum_{s=k}^\infty \delta_s \tag{31}$$

Now, from (31), analogously to (26), we deduce

$$\left|L^{-1}\left(\frac{\varphi^2}{2\rho v_k^2} - \frac{\varphi^2}{2\rho v^2}\right)\right| \leqslant m_{12}\left\|\sum_{s=k}^\infty \delta_s\right\|_{L_{2,\rho}} \leqslant mq^{k-1}\|\alpha_1\|_{L_2^*} \tag{32}$$

i.e. Equation (4) is obtained from (29) in the limit as $k \to \infty$.

The estimates (3) for v' and v'' can be found from (4) by the use of (30). The theorem is thus proved.

Suppose $\varphi(1) \neq 0$. Then, for the function $u = dw/d\rho$ where w is the deflection, we have

$$u = -\frac{\varphi(\rho)}{v} = O\left((1-\rho)^{-2/3}\right)$$

Mechanically, this means that in this case equilibrium of the membrane is not possible. It was therefore natural to consider the case [2] when the resultant of the system of forces acting on the membrane was zero, i.e. $\varphi(1) = 0$.

With this condition the solution proves to be smoother and has two continuous derivatives. The appropriate theorem was formulated in [2]. Its proof coincides almost exactly with the proof given above. The difference lies in the norms introduced by the formulas (15) and (16) where the weight $(1-\rho^2)$ and $(1-\rho)^{-2}$ should be taken as $(1-\rho)$ and $(1-\rho)^{-1}$ respectively.

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